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Integrable Classical Systems in Higher Dimensions

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A general method for the construction of the second constant of motion (up to second order) for higher-dimensional classical systems is carried out. Correspondingly, the first- and the second-order potential equations are obtained whose solutions can directly provide the integrable systems.

1. INTRODUCTION

In recent years, there has been considerable interest in the study of time-dependent (Günther and Leach, 1977; Leach, 1981; Mishra *et al.*, 1984; Kaushal *et al.*, 1984; Mishra, 1985) and time-independent (Hall, 1983; Holt, 1982; Kaushal *et al.*, 1985; Kaushal and Mishra, 1986) integrable classical dynamical systems in one and two dimensions. Construction of invariants for such systems facilitates the solution of nonlinear differential equations. There exists at present no general method for testing the integrability of a given dynamical system. However, the Painlevé method (Dorizzi *et al.*, 1983) detects the integrability of a dynamical system with the use of singularity analysis and direct calculation of the second integral of motion. Whittaker (1927) first investigated the problem of the construction of an invariant other than the total energy, which goes by the name of the second constant of motion. His studies were, however, restricted to the invariant of first or second order in momenta. Although there have been several attempts (Günther and Leach, 1977; Leach, 1981; Mishra *et al.*, 1984; Kaushal *et al.*, 1984; Mishra, 1985; Hall, 1983; Holt, 1982; Kaushal *et al.*, 1985; Kaushal and Mishra, 1986; Dorizzi *et al.*, 1985; Whittaker, 1927; Fokas and Lagerstrom, 1980; Inozemtsev, 1983) in recent years to construct the second- and

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higher-order invariants in two dimensions, not much effort has been made to obtain the invariants to (higher) three dimensions. Grammaticos *et al.* (1985) discussed a method of constructing N-dimensional integrable systems starting from the two-dimensional ones. They further carried out the singularity analysis of the equations of motion, which led them to systems exhibiting the Painlevé property, i.e., the only movable singularities of the solutions in the complex time plane were assumed to be pole type. These results are also discussed for different cases of N-dimensional systems by Lakshmanan and Sahadevan (1984). In some cases, no doubt, the system is found to be integrable just by accident.

Dorizzi *et al.* (1986) investigated the existence of integrable systems in three dimensions in which they reduced three-dimensional systems to two-dimensional ones using cylindrical symmetry and solved for quartic potentials. Our method is quite different from that of Dorizzi *et al.* (1986). In this investigation, we present a recipe for the construction of certain potentials and corresponding invariants of a particular type for three-dimensional time-independent classical systems. A general mathematical formulation is described in Section 2. In Section 3, we show, in addition to other examples, that a well-known potential (Fokas and Lagerstrom, 1980; Inozemtsev, 1983) of the type $(x_1x_2)^{-2/3}$ (which admits a third-order invariant in two dimensions) admits a second-order invariant in three dimensions. In particular, our method is general and we construct the invariants without the need for reducing the dimensions. Section 4 contains concluding remarks.

2. THE METHOD

We consider a dynamical system described by the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3) \tag{1}$$

with the concomitant equations of motion

$$\ddot{x}_1 = -\frac{\partial V}{\partial x_1}, \qquad \ddot{x}_2 = -\frac{\partial V}{\partial x_2}, \qquad \ddot{x}_3 = -\frac{\partial V}{\partial x_3} \tag{2}$$

Let us assume the existence of the second constant of motion (called invariant hereafter) I, up to second order in momenta in a general form as

$$I = a_0 + a_i\xi_i + \frac{1}{2!}\, a_{ij}\xi_i\xi_j \tag{3}$$

where

$$i, j = 1, 2, 3, \qquad \xi_1 = \dot{x}_1, \qquad \xi_2 = \dot{x}_2, \qquad \xi_3 = \dot{x}_3 \tag{4}$$

and the coefficients a_0, a_i, and a_{ij} are functions of x_1, x_2, and x_3 only. These coefficients are symmetric with respect to only interchange of their indices. Here we also assume that the invariant contains either only even powers of momenta or only odd powers of momenta (Hall, 1983; Holt, 1982; Kaushal *et al.*, 1985; Kaushal and Mishra, 1986; Hietarinta, 1987). The invariance of I implies $dI/dt = 0$, and using (3), we get

$$\frac{dI}{dt} = 0 = a_{0,i}\xi_i + a_{i,j}\xi_i\xi_j + a_i\dot{\xi}_i + \tfrac{1}{2}a_{ij,k}\xi_i\xi_j\xi_k + \tfrac{1}{2}a_{ij}(\dot{\xi}_i\xi_j + \xi_i\dot{\xi}_j) \tag{5}$$

After accounting for the proper symmetrization of the coefficients and noting that (5) must hold identically in ξ's, we obtain the following relations:

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0 \tag{6}$$

$$a_{i,j} + a_{j,i} = 0 \tag{7}$$

$$a_{0,i} + a_{ij}\dot{\xi}_j = 0 \tag{8}$$

$$a_i\dot{\xi}_i = 0 \tag{9}$$

Equations (7) and (9) after using (4) yield the following set of partial differential equations:

$$\frac{\partial a_1}{\partial x_1} = 0 \tag{10}$$

$$\frac{\partial a_2}{\partial x_2} = 0 \tag{11}$$

$$\frac{\partial a_3}{\partial x_3} = 0 \tag{12}$$

$$\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_1} = 0 \tag{13}$$

$$\frac{\partial a_1}{\partial x_3} + \frac{\partial a_3}{\partial x_1} = 0 \tag{14}$$

$$\frac{\partial a_2}{\partial x_3} + \frac{\partial a_3}{\partial x_2} = 0 \tag{15}$$

$$a_1\ddot{x}_1 + a_2\ddot{x}_2 + a_3\ddot{x}_3 = 0 \tag{16}$$

whereas equations (6) and (8) yield

$$\frac{\partial a_{11}}{\partial x_1} = 0 \tag{17}$$

$$\frac{\partial a_{11}}{\partial x_2}+2\frac{\partial a_{12}}{\partial x_1}=0 \tag{18}$$

$$\frac{\partial a_{11}}{\partial x_3}+2\frac{\partial a_{13}}{\partial x_1}=0 \tag{19}$$

$$\frac{\partial a_{22}}{\partial x_1}+2\frac{\partial a_{12}}{\partial x_2}=0 \tag{20}$$

$$\frac{\partial a_{12}}{\partial x_3}+\frac{\partial a_{23}}{\partial x_1}+\frac{\partial a_{13}}{\partial x_2}=0 \tag{21}$$

$$\frac{\partial a_{33}}{\partial x_1}+2\frac{\partial a_{13}}{\partial x_3}=0 \tag{22}$$

$$\frac{\partial a_{22}}{\partial x_2}=0 \tag{23}$$

$$\frac{\partial a_{22}}{\partial x_3}+2\frac{\partial a_{23}}{\partial x_2}=0 \tag{24}$$

$$\frac{\partial a_{33}}{\partial x_2}+2\frac{\partial a_{23}}{\partial x_3}=0 \tag{25}$$

$$\frac{\partial a_{33}}{\partial x_3}=0 \tag{26}$$

$$\frac{\partial a_0}{\partial x_1}=a_{11}\frac{\partial V}{\partial x_1}+a_{12}\frac{\partial V}{\partial x_2}+a_{13}\frac{\partial V}{\partial x_3} \tag{27}$$

$$\frac{\partial a_0}{\partial x_2}=a_{12}\frac{\partial V}{\partial x_1}+a_{22}\frac{\partial V}{\partial x_2}+a_{23}\frac{\partial V}{\partial x_3} \tag{28}$$

$$\frac{\partial a_0}{\partial x_3}=a_{13}\frac{\partial V}{\partial x_1}+a_{23}\frac{\partial V}{\partial x_2}+a_{33}\frac{\partial V}{\partial x_3} \tag{29}$$

Now we present the solutions of these equations for determining various coefficients. From equations (10)-(12) we can write

$$a_1=f_1(x_2, x_3), \qquad a_2=f_2(x_1, x_3), \qquad a_3=f_3(x_1, x_2)$$

To solve equations (13) and (15), we differentiate them wrt x_3 and x_1, respectively, and obtain

$$\frac{\partial}{\partial x_2}\left(\frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1}\right)=0 \tag{30}$$

Using (14), this equation leads to

$$a_1 = \bar{g}_1(x_2) + g_2(x_3) \tag{31}$$

Similarly, a_2 and a_3 can be obtained in the form

$$a_2 = \bar{g}_3(x_3) + g_4(x_1) \tag{32}$$

$$a_3 = \bar{g}_5(x_1) + g_6(x_2) \tag{33}$$

where $\bar{g}_i$ and g_i are arbitrary functions of their corresponding arguments. In order to determine them, we use (31) and (32) in (13) to get

$$\frac{d\bar{g}_1(x_2)}{dx_2} = -\frac{dg_4(x_1)}{dx_1} = \text{const} \quad (\text{say } C_1) \tag{34}$$

which implies that

$$\bar{g}_1 = C_1 x_2 + C_2 \tag{35}$$

$$g_4 = -C_1 x_1 + C_3 \tag{36}$$

where C_1 is the separation constant and C_2, C_3 are integration constants. Similarly, we can find the values of g_2, $\bar{g}_3$, $\bar{g}_5$, and g_6 as

$$g_2 = -C_4 x_3 + C_6 \tag{37}$$

$$\bar{g}_3 = C_7 x_3 + C_8 \tag{38}$$

$$\bar{g}_5 = C_4 x_1 + C_5 \tag{39}$$

$$g_6 = -C_7 x_2 + C_9 \tag{40}$$

where C_4, C_7 are the separation constants and C_5, C_6, C_8, and C_9 are the integration constants.

Substituting these values of $\bar{g}_1$, g_2, $\bar{g}_3$, g_4, $\bar{g}_5$, and g_6 into equations (31)-(33) we get

$$a_1 = C_1 x_2 - C_4 x_3 + C_2' \tag{41}$$

$$a_2 = C_7 x_3 - C_1 x_1 + C_3' \tag{42}$$

$$a_3 = C_4 x_1 - C_7 x_2 + C_5' \tag{43}$$

where $C_2' = C_2 + C_6$, $C_3' = C_3 + C_8$, and $C_5' = C_5 + C_9$.

Again using equation (16) along with equation (2), we arrive at the "potential equation,"

$$(C_1 x_2 - C_4 x_3 + C_2') \frac{\partial V}{\partial x_1} + (C_7 x_3 - C_1 x_1 + C_3') \frac{\partial V}{\partial x_2} + (C_4 x_1 - C_7 x_2 + C_5') \frac{\partial V}{\partial x_3} = 0 \tag{44}$$

The solution of this equation will provide directly the first-order invariants for three-dimensional systems. Now we solve equations (17)-(29). Equations (17), (23), and (26) clearly imply that

$$a_{11} = a_{11}(x_2, x_3) \tag{45}$$

$$a_{22} = a_{22}(x_1, x_3) \tag{46}$$

$$a_{33} = a_{33}(x_1, x_2) \tag{47}$$

Using these results, equations (18) and (20) will yield two expressions for a_{12} respectively as

$$a_{12} = h_1(x_2, x_3)x_1 + h_2(x_2, x_3) \tag{48}$$

and

$$a_{12} = h_3(x_1, x_3)x_2 + h_4(x_1, x_3) \tag{49}$$

where h_i are arbitrary functions of their arguments. In order for these two expressions for a_{12} to be the same, one has to take recourse to certain plausible choices on the functions h_i which evidently are constrained by the functional forms of their arguments. These considerations restrict the possible choice of the h_i to the following identifications:

$$\begin{aligned} h_1 &= S_3x_2 + \alpha, \qquad & h_2 &= \alpha x_2 + K \\ h_3 &= S_3x_1 + \alpha, \qquad & h_4 &= \alpha x_1 + K \end{aligned} \tag{50}$$

where S_3 is a dimensionless constant and α and K are introduced to account for the dimensional consistency of the equations. With these prescriptions, the resulting expression for a_{12} has the form

$$a_{12} = S_3x_1x_2 + \alpha(x_1 + x_2) + K \tag{51}$$

which contains, in addition, a linear term proportional to α.

Proceeding exactly similarly, we can write down expressions for a_{13} and a_{23} as follows:

$$a_{13} = S_2x_1x_3 + \alpha(x_1 + x_3) + K \tag{52}$$

$$a_{23} = S_1x_2x_3 + \alpha(x_2 + x_3) + K \tag{53}$$

Substitution of equation (51) in (18) and (19) immediately gives

$$a_{11} = -S_3x_2^2 - S_2x_3^2 - 2\alpha(x_2 + x_3) + \text{const} \tag{54}$$

Similar expressions for a_{22} and a_{33} can be obtained by substituting equation

(52) in (20) and (24), and equation (52) in (22) and (25). Thus, we have

$$a_{22} = -S_3 x_1^2 - S_1 x_3^2 - 2\alpha(x_1 + x_3) + \text{const} \tag{55}$$

$$a_{33} = -S_2 x_1^2 - S_1 x_2^2 - 2\alpha(x_1 + x_2) + \text{const} \tag{56}$$

On eliminating a_0 from equations (27) and (28), we get

$$\begin{aligned} &\frac{\partial V}{\partial x_1}\left(\frac{\partial a_{11}}{\partial x_2} - \frac{\partial a_{12}}{\partial x_1}\right) + \frac{\partial^2 V}{\partial x_1 \partial x_2}(a_{11} - a_{22}) \\ &+ \frac{\partial V}{\partial x_2}\left(\frac{\partial a_{12}}{\partial x_2} - \frac{\partial a_{22}}{\partial x_1}\right) + a_{12}\left(\frac{\partial^2 V}{\partial x_2^2} - \frac{\partial^2 V}{\partial x_1^2}\right) \\ &+ \frac{\partial V}{\partial x_3}\left(\frac{\partial a_{13}}{\partial x_2} - \frac{\partial a_{23}}{\partial x_1}\right) + \left(a_{13}\frac{\partial^2 V}{\partial x_2 \partial x_3} - a_{23}\frac{\partial^2 V}{\partial x_1 \partial x_3}\right) = 0 \end{aligned} \tag{57}$$

Substituting the values of a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, and a_{23} in (57), we get the potential equation

$$\begin{aligned} &3\left[(S_3 x_1 + \alpha)\frac{\partial V}{\partial x_2} - (S_3 x_2 + \alpha)\frac{\partial V}{\partial x_1}\right] \\ &+ [S_3(x_1^2 - x_2^2) + x_3^2(S_1 - S_2) + 2\alpha(x_1 - x_2)]\frac{\partial^2 V}{\partial x_1 \partial x_2} \\ &+ [S_3 x_1 x_2 + \alpha(x_1 + x_2) + K]\left(\frac{\partial^2 V}{\partial x_2^2} - \frac{\partial^2 V}{\partial x_1^2}\right) + [S_2 x_1 x_3 + \alpha(x_1 + x_3) + K] \\ &\times\left(\frac{\partial^2 V}{\partial x_2 \partial x_3}\right) - [S_1 x_2 x_3 + \alpha(x_2 + x_3) + K]\left(\frac{\partial^2 V}{\partial x_1 \partial x_3}\right) = 0 \end{aligned} \tag{58}$$

Similarly, using equations (28) and (29, (27) and (29), and substituting a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, and a_{23}, we get the following potential equations:

$$\begin{aligned} &3\left[(S_1 x_2 + \alpha)\frac{\partial V}{\partial x_3} - (S_1 x_3 + \alpha)\frac{\partial V}{\partial x_2}\right] \\ &+ [S_1(x_2^2 + x_3^2) + x_1^2(S_2 - S_3) + 2\alpha(x_2 - x_3)] \\ &\times\frac{\partial^2 V}{\partial x_2 x_3} + [S_1 x_2 x_3 + \alpha(x_2 + x_3) + K]\left(\frac{\partial^2 V}{\partial x_3^2} - \frac{\partial^2 V}{\partial x_2^2}\right) \\ &+ [S_3 x_1 x_2 + \alpha(x_1 + x_2) + K] \\ &\times\frac{\partial^2 V}{\partial x_1 \partial x_3} - [S_2 x_1 x_3 + \alpha(x_1 + x_3) + K]\frac{\partial^2 V}{\partial x_1 \partial x_2} = 0 \end{aligned} \tag{59}$$

and

$$3\left[(S_2x_1+\alpha)\frac{\partial V}{\partial x_3}-(S_2x_3+\alpha)\frac{\partial V}{\partial x_1}\right]+[S_2(x_1^2-x_3^2)+x_2^2(S_1-S_3)$$

$$+2\alpha(x_1-x_2)]\frac{\partial^2 V}{\partial x_1 \partial x_3}+[S_3x_1x_2+\alpha(x_1+x_2)+K]\frac{\partial^2 V}{\partial x_2\partial x_3}$$

$$+[S_2x_1x_3+\alpha(x_1+x_3)+K]\left(\frac{\partial^2 V}{\partial x_3^2}-\frac{\partial^2 V}{\partial x_1^2}\right)$$

$$-[S_1x_2x_3+\alpha(x_2+x_3)+K]\frac{\partial^2 V}{\partial x_1\partial x_2}=0 \tag{60}$$

In principle, the solutions of these equations [(58)-(60)] will directly provide the systems admitting second-order invariants. However, the solutions of these equations, in general, are rather involved. In the following section, we solve these potential equations for some specific forms of V.

3. ILLUSTRATIVE EXAMPLES

1. The potential V is separable in x_i and has the form

$$V(x_1, x_2, x_3)=x_1^m+x_2^n+x_3^l \tag{61}$$

Substituting this form of V in the potential equations (58)-(60), we find that all these equations provide a solution

$$\alpha=K=0 \qquad \text{and} \qquad m, n, l=-2 \tag{62}$$

Thus, the potential becomes

$$V(x_1, x_2, x_3)=\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2} \tag{63}$$

Using (63) in equations (27)-(29) and demanding the compatibility of the solutions, we find

$$a_0=-\left[\frac{1}{x_1^2}(S_3x_2^2+S_2x_3^2)+\frac{1}{x_2^2}(S_3x_1^2+S_1x_3^2)+\frac{1}{x_3^2}(S_2x_1^2+S_1x_2^2)\right] \tag{64}$$

and the invariant (3) corresponding to the potential (63) now becomes

$$I=-\tfrac{1}{2}[S_3(\dot{x}_1x_2-x_1\dot{x}_2)^2+S_2(\dot{x}_1x_3-x_1\dot{x}_3)^2+S_1(\dot{x}_2x_3-x_2\dot{x}_3)^2]$$

$$-\left[\frac{1}{x_1^2}(S_3x_2^2+S_2x_3^2)+\frac{1}{x_2^2}(S_3x_1^2+S_1x_3^2)+\frac{1}{x_3^2}(S_2x_1^2+S_1x_2^2)\right] \tag{65}$$

2. Let V be separable in x_i having the form

$$V(x_1, x_2, x_3) = x_1^m x_2^n x_3^l \tag{66}$$

Now substituting (66) in the potential equations (58)-(60), we find

$$\alpha = K = 0, \qquad S_1 = S_2 = S_3 = S \tag{67}$$

and

$$m + n + l = -2 \tag{68}$$

Using (67), equations (51)-(56) become

$$a_{11} = -S(x_2^2 + x_3^2), \qquad a_{22} = -S(x_1^2 + x_3^2), \qquad a_{33} = -S(x_1^2 + x_2^2)$$

$$a_{12} = Sx_1x_2, \qquad a_{13} = Sx_1x_3, \qquad a_{23} = Sx_2x_3$$

In fact, condition (68) yields a large class of potentials. However, we see that a unique solution is indeed possible from the expressions of a_0 which are found after using (66) in (27)-(29),

$$\begin{aligned} a_0 &= -S(x_2^2 + x_3^2)x_1^m x_2^n x_3^l + \frac{S}{m+2} x_1^{m+2} x_2^n x_3^l (n+l) + K_1(x_2, x_3) \\ a_0 &= -S(x_1^2 + x_3^2)x_1^m x_2^n x_3^l + \frac{S}{n+2} x_1^m x_2^{n+2} x_3^l (m+l) + K_2(x_1, x_3) \\ a_0 &= -S(x_1^2 + x_2^2)x_1^m x_2^n x_3^l + \frac{S}{l+2} x_1^m x_2^n x_3^{l+2} (m+n) + K_3(x_1, x_3) \end{aligned} \tag{69}$$

where K_1, K_2, K_3 are arbitrary functions of their arguments.

In this case we obtain three different expressions for a_0 which limit the freedom to a unique choice, i.e., $K_1 = K_2 = K_3 = 0$ and $m = n = l = -2/3$, which also agrees with equation (68). Finally, the expression for a_0 becomes

$$a_0 = -S(x_1^2 + x_2^2 + x_3^2)(x_1x_2x_3)^{-2/3} \tag{70}$$

and the invariant (3) corresponding to the potential $(x_1x_2x_3)^{-2/3}$ becomes

$$\begin{aligned} I = -S\{\tfrac{1}{2}[(\dot{x}_1x_2 - x_1\dot{x}_2)^2 + (\dot{x}_1x_3 - x_1\dot{x}_3)^2 \\ + (\dot{x}_2x_3 - x_2\dot{x}_3)^2] + (x_1^2 + x_2^2 + x_3^2)(x_1x_2x_3)^{-2/3}\} \end{aligned} \tag{71}$$

3. Let the potential be in the spherically symmetric form, viz.,

$$V = \beta(x_1^2 + x_2^2 + x_3^2), \qquad \beta = \text{const} \tag{72}$$

The expression for a_0 is obtained after substituting (72) in equations (27)-(29) and using equations (51)-(56). The result is

$$a_0 = \tfrac{4}{3}\beta K(x_1x_2 + x_2x_3 + x_1x_3) \tag{73}$$

The invariant (3) corresponding to the potential (72) is explicitly given by

$$\begin{aligned} I = {} & \tfrac{4}{3}\beta K(x_1x_2+x_2x_3+x_1x_3) \\ & + S_1[x_2x_3\dot{x}_2x_3 - \tfrac{1}{2}(x_3^2\dot{x}_2^2 + x_2^2\dot{x}_3^2)] \\ & + S_2[x_1x_3\dot{x}_1\dot{x}_3 - \tfrac{1}{2}(x_1^2\dot{x}_3^2 + x_3^2\dot{x}_1^2)] \\ & + S_3[x_1x_2\dot{x}_1\dot{x}_2 - \tfrac{1}{2}(x_1^2\dot{x}_2^2 + x_2^2\dot{x}_1^2)] \\ & + K(\dot{x}_1\dot{x}_2 + \dot{x}_2\dot{x}_3 + \dot{x}_1\dot{x}_3) \end{aligned} \tag{74}$$

4. Let the potential be given by

$$V = x_1^2 + x_2^2 + x_3^{-2} \tag{75}$$

Proceeding exactly as in the previous cases, the expression for a_0 turns out to be

$$a_0 = -S_1(x_1^2+x_2^2)(x_3^2+x_3^{-2}) \tag{76}$$

and the corresponding invariant has the form

$$\begin{aligned} I = {} & -S_1[(x_1^2+x_2^2)(x_3^2+x_3^{-2}) + \tfrac{1}{2}(x_3\dot{x}_1 - x_1\dot{x}_3)^2 + \tfrac{1}{2}(x_2\dot{x}_3 - x_3\dot{x}_2)^2] \\ & - S_3[\tfrac{1}{2}(x_2\dot{x}_1 - x_1\dot{x}_2)^2] \end{aligned} \tag{77}$$

4. CONCLUSIONS

In conclusion, a few remarks seem appropriate. The method outlined in the present investigation furnishes a general structure for the potential equations in three dimensions. The solutions of these equations are capable of providing integrable systems admitting second-order invariants. A set of four systems is examined in the context of our framework which admit second-order invariants. One of these systems, described by the potential of example 2, seems to be an interesting case particularly because it represents a generalization of the well-known Fokas potential to three dimensions.

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